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Determination of Modal Residues and Residual Flexibility for Time-Domain System Realization

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A linear least-squares procedure for the determination of modal residues using time-domain system realization theory is presented. The present procedure is intended to complement existing techniques for time-domain system identification and is shown to be theoretically equivalent to residue determination in realization algorithms such as the eigensystem realization algorithm and *Q*-Markov covariance equivalent realization method. However, isolating the optimal residue estimation problem from the general realization problem affords several alternative strategies as compared to standard realization algorithms for structural dynamics identification. Primary among these are alternative techniques for handling data sets with large numbers of sensors using small numbers of reference point responses and the inclusion of terms that accurately model the effects of residual flexibility. The accuracy and efficiency of the present realization theory-based procedure is demonstrated for both simulated and experimental data.

I. Introduction

RESEARCH in structural identification in recent years has lead to a proliferation of algorithms based on system realization theory.[1] These system identification techniques[2–4] have important direct applications to structural control, such as identification and order reduction of input–output models for robust control and adaptive on-line identification for nonlinear systems control. These algorithms all identify a model by minimizing some measure of the difference between the measured and reconstructed discrete-time impulse response functions, heretofore referred to as Markov parameters. In contrast to many classical modal identification techniques, the system realization algorithms are time-domain techniques and are generally applicable to multiple-input/multiple-output measured data systems.

These algorithms have arguably attracted the most attention for their use in modal test data analysis and reduction for identification of structural parameters.[5,6] There are a number of reasons for this. First, these methods are fairly simple to understand and implement, requiring only standard matrix manipulation and numerical analysis functions such as those available in MatLab.[7] Second, these methods are founded on sampled data systems theory, which is directly applicable to inexpensive microprocessor-based data acquisition systems. Finally, system realization theory offered a simplification of the modal identification process by providing a clear indication (at least ideally) of dynamic order and by unifying the pole identification and residue estimation problems into a single step analysis. In other words, these methods were powerful tools at the right time. They provided a cookbook approach to structural identification for engineers, including those unfamiliar with existing modal parameter identification methods and research.

These popular realization algorithms have, however, lacked the practical capabilities inherent in many standard modal identification software packages. Although these standard software packages use some multiple reference time-domain identification techniques, such as Polyreference,[8] they also feature separate treatment of pole identification and residue estimation and the capability to estimate residual flexibility and inertia, which improve model reconstructions. By contrast, eigensystem realization algorithm (ERA) and other system realization theory-based techniques[9] identify simultaneously the poles and residues in a unified model and do not generally provide for the modeling of residual effects. In the original formulations of these system realization techniques, the use of a large number of time steps quickly led to large data matrices that could not be efficiently handled. Furthermore, many researchers have noted problems in achieving highly accurate reconstructions of some types of modal data using discrete-time-domain realization algorithms, which has led, among other things, to the development of frequency-domain-based realization techniques[10,11] and residue re-estimation.[12,13]

The purpose of this paper is to develop additional practical capabilities for modern time-domain realization-based algorithms (such as ERA) through system realization theory and terminology. As such, we intend the present paper to provide a natural complement to existing system realization literature. Our approach is based on a time-domain least-squares estimation of the modal residues and residual flexibility, given a prior identification of the pole information, i.e., frequencies and damping rates, and the modal participation factors of the system inputs. That is, for the discrete-time state-space model

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k) + Du(k) \quad (1)$$

our approach determines C and D given that A and B have been identified in a prior analysis, using perhaps a subset of the measured response functions. We will show how this estimation is consistent with, and related to, the residue estimation implicit in existing system realization algorithms. The present procedure also provides a purely time-domain alternative to the approach of re-estimating of the mode shapes in the frequency domain using the measured frequency response functions.[12,13]

To this end, the paper is organized as follows. In Sec. II, the time-domain-based system realization theory and procedure is presented. For reasons of clarity and conciseness, only the ERA procedure is detailed. In Sec. III, a procedure for optimally computing the mode shape matrix C using a linear least-squares solution with A and B from Eq. (1) is detailed using system realization theory, and its relationship to the ERA computation of C is examined. In Sec. IV, the present mode shape estimation procedure is utilized to develop three useful generalizations of ERA for identification of structural dynamic models. Finally, Sec. V applies the present procedure to

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a very realistic simulated data example and to experimental data. Conclusions are offered in Sec. VI.

II. Review of Time-Domain System Realization Procedure

We begin by presenting the governing equations of motion for structural dynamics in their usual forms. The response of a structure to a set of forces or inputs $\boldsymbol{u}(t)$ is usually modeled as a spatially discretized second-order matrix differential equation of the form

$$M\ddot{q} + D\dot{q} + Kq = \hat{B}u \tag{2}$$

where M is the mass matrix, D is the damping matrix, K is the stiffness matrix, and $\hat{B}$ is the force influence matrix. The vector $q(t)$ includes all of the physical degrees of freedom (DOFs) of the model. If we define the n associated normal modes Φ_n of Eq. (1) according to

$$K\Phi_n = M\Phi_n\Omega \tag{3}$$

$$\begin{gathered}\Phi_n^T K\Phi_n = \Omega = \{\omega_{ni}^2, i = 1, \ldots, n\} \\ \Phi_n^T M\Phi_n = I_{n\times n}, \qquad \Phi_n^T D\Phi_n = \Xi\end{gathered} \tag{4}$$

then the structural model can be placed into the first-order modal state-space form

$$\dot{x}_\eta = A_\eta x_\eta + Bu, \qquad y = C_\eta x_\eta + Du \tag{5}$$

in which $y(t)$ is a vector of measured responses and the modal state-space matrices are given by

$$\begin{gathered}A_\eta = \begin{bmatrix} 0 & I \\ -\Omega & -\Xi \end{bmatrix}, \qquad B_\eta = \begin{bmatrix} 0 \\ \Phi_n^T\hat{B} \end{bmatrix} \\ C_\eta = [H_d \quad 0] + [H_v \quad 0]A_\eta + [H_a \quad 0]A_\eta^2\end{gathered} \tag{6}$$

in which H_d, H_v, and H_a are the output displacement, velocity, and acceleration location influence arrays, respectively.

Because all experimental vibration data is sampled in time, all time-domain linear system realization procedures begin from the presumption that a finite-order discrete state-space model of the system exists of the form

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k) + Du(k) \tag{7}$$

in which k is the time sample index. The procedure by which Eq. (5) is sampled to lead to Eq. (7) must be done carefully to avoid ill conditioning due to the transformation from the continuous s plane to the discrete z plane. Likewise, the transformation from a realized model of the form of Eq. (7) back to the continuous representation of Eq. (5) requires careful eigenrotation and mass normalization, as described in Ref. 14.

When the model of Eq. (7) is used as a predictor, the arbitrary response to an input $u(k)$ is given by

$$y(k) = \sum_{i=1}^{k} M(k-i)u(i), \qquad 1 \le k < \infty \tag{8}$$

in which the system Markov parameters $M(k)$ are related to the state-space matrices by

$$M(k) = \begin{cases} D & k = 0 \\ CA^{k-1}B & k > 0 \end{cases} \tag{9}$$

All state-space time-domain realization methods attempt to find the state space matrices A, B, C, and D from measurements of the sequence $M(k)$. This is the process known as system realization.

The essential considerations in system realization are the selection of the model order (it is presumed that the model form is correct) and the determination of the state-space parameters from a minimization of some prediction error. For ERA, the prediction error is defined in terms of a Hankel matrix of the Markov parameters, as defined by

$$H_{rs}(k) = \begin{bmatrix} M(k+1) & M(k+2) & \cdots & M(k+s) \\ M(k+2) & M(k+3) & \cdots & M(k+s+1) \\ \vdots & \vdots & \cdots & \vdots \\ M(k+r) & M(k+r+1) & \cdots & M(k+r+s-1) \end{bmatrix} \tag{10}$$

The ERA realization finds the linear least-squares solution to minimize the error in the shift in the Hankel matrix of the system model and the data according to

$$H_{rs}(k-1) = V_r A^{k-1} W_s \tag{11}$$

in which

$$V_r = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}, \qquad W_s = [B \quad AB \quad \cdots \quad A^{s-1}B] \tag{12}$$

If the Hankel matrix is formed from the data, then the factors V_r and W_s are obtained from a singular value decomposition (SVD) of the 0th Hankel matrix according to

$$H_{rs}(0) = P_r S_{rs} Q_s^T, \qquad V_r = P_r S_{rs}^{\frac{1}{2}}, \qquad W_s = S_{rs}^{\frac{1}{2}} Q_s^T \tag{13}$$

The model order is selected (in principle) by examining the numerical rank of $H_{rs}(0)$. From this, the system realization problem is solved by

$$\begin{gathered}A = S_{rs}^{-\frac{1}{2}} P_r^T H_{rs}(1) Q_s S_{rs}^{-\frac{1}{2}}, \qquad B = S_{rs}^{\frac{1}{2}} Q_s^T(1{:}n_x, 1{:}n_u) \\ C = P_r(1{:}n_y, 1{:}n_x) S_{rs}^{\frac{1}{2}}, \qquad D = M(0)\end{gathered} \tag{14}$$

Reference 15 discusses the computationally more efficient approach of factoring $H_{rs}(0)H_{rs}^T(0)$ instead of $H_{rs}(0)$ to obtain the factors of Eq. (13). In this case, it is more computationally efficient to calculate the factors using a symmetric eigensolver in place of the SVD. By only computing the largest n_x eigenvalues and vectors of the Hankel matrix product, it is possible to determine realizations using very large values of r and s without calculating the entire spectral decomposition.

III. Time-Domain Residue Estimation

Using the terminology consistent with system realization theory and outlined in Sec. II, we now develop the time-domain residue estimation algorithm. In the equations that follow, the tilde ($\tilde{}$) applied to the controllability and observability matrices or the Markov parameters is used to represent the estimate or reconstruction of those quantities from the realized model, whereas the quantities without the tilde are understood to be the exact values or measured values of those quantities.

A. Least-Squares Solution for Residues

Suppose the state-space matrices A and B have been determined from a data set using, for example, ERA or Q-Markov COVER. Then, using Eq. (9), we have

$$\begin{gathered}CB = M(1) \\ CAB = M(2) \\ \vdots \\ CA^{s-1}B = M(s)\end{gathered} \tag{15}$$

Hence,

$$C\tilde{W}_s = \{M(1{:}s)\} \tag{16}$$

where $\tilde{W}_s$ is the estimate of the controllability matrix as reconstructed from the model realization and $\{M(1{:}s)\}$ is the columnwise arrangement of the first s measured Markov parameters, viz.,

$$\begin{gathered}\tilde{W}_s = [B \quad AB \quad \cdots \quad A^{s-1}B] \\ \{M(1{:}s)\} = [M(1) \quad M(2) \quad \cdots \quad M(s)]\end{gathered} \tag{17}$$

The least-squares solution for C using Eq. (16) is given by

$$C = \{M(1{:}s)\}\tilde{W}_s^+ = \{M(1{:}s)\}\tilde{W}_s^T\left[\tilde{W}_s\tilde{W}_s^{T}\right]^{-1} \quad (18)$$

The solution is well defined as long as $s > n$, where n is the system order (dimension of A). Note that the solution for D, that is,

$$D = M(0) \quad (19)$$

holds under the present theory, so long as additional terms, such as residual flexibility, are not added to the problem.

The implementation of Eq. (18) is equally straightforward. This is because we can utilize the SVD of $H_{rs}(0)$ used in the earlier analysis to realize A and B to determine $\tilde{W}_s^+$, viz.,

$$\tilde{W}_s^+ = Q_s S_{rs}^{-\frac{1}{2}} \quad (20)$$

Thus,

$$C = \{M(1{:}s)\}Q_s S_{rs}^{-\frac{1}{2}} \quad (21)$$

B. Relationship to Residue Estimation in ERA

As reviewed in Sec. II, in ERA the solution for C is given by

$$C = E_l\tilde{V}_r = E_l P_r S_{rs}^{\frac{1}{2}} \quad (22)$$

where $E_l = [I_{l\times l}\ 0\ \cdots\ 0]$ and $\tilde{V}_r$ is the estimated observability matrix, which is realized from the SVD of the measured Hankel matrix $H_{rs}(0)$, viz.,

$$H_{rs}(0) \approx \tilde{H}_{rs}(0) = \tilde{V}_r\tilde{W}_s = P_r S_{rs} Q_s^T \quad (23)$$

Then, from Eq. (22) we have

$$C = E_l\tilde{V}_r\tilde{W}_s\tilde{W}_s^+ = E_l\tilde{H}_{rs}(0)\tilde{W}_s^+ = \{\tilde{M}(1{:}s)\}\tilde{W}_s^+ \quad (24)$$

Thus, comparing Eqs. (18) and (24), the new least-squares solution for C is fully consistent with the system realization theory-based residue determination. The fundamental distinction is that the ERA solution for C in Eq. (24) is optimal for the realized Markov parameters, that is, the approximated Markov parameters as expressed by the realized or approximated Hankel matrix $\tilde{H}_{rs}(0)$, whereas the least-squares solution in Eq. (18) is optimal for the actual measured Markov parameters.

IV. Algorithms Based on Present Theory

A. ERA Using Reference Point Responses (ERA-RP)

The residue estimation algorithm presented in Sec. III leads to a very useful generalization of ERA for structural dynamics identification. Because the modes that are identifiable from the data are limited to those that are disturbable from the system inputs, it is only necessary to include a small number of reference point responses from which the same modes are observable. In the case of structural dynamics when the system is reciprocal, i.e., symmetric mass, stiffness, and damping properties, the logical sensor complements are driving point measurements, that is, sensors collocated with the system input DOFs.

The prime advantage of this approach is that it enables the use of longer data records for the same Hankel matrix dimension, or allows the reduction of the Hankel matrix dimension to increase overall computational efficiency. The use of longer data records is important for obtaining accurate and consistent frequency and damping estimates from real data. Reducing the size of the Hankel matrix is also important because the major computational overhead in the system realization procedure is strongly dictated by the minimum dimension of the matrix.

For example, suppose a typical modal test of a complex structure is performed for the purpose of characterizing the normal modes. If the test is measured using 100 accelerometers and 3 force inputs, an ERA analysis might utilize Hankel block dimensions of $r = 50$ and $s = 2000$, leading to a Hankel matrix of size 5000×6000. On the other hand, a reference point ERA analysis might instead use $r = 500$ and $s = 1600$, for a Hankel matrix of size 1500×4800. In the latter case, the length of the Markov sequence actually used in the Hankel matrix is slightly greater than in the ERA analysis, but the minimum matrix dimension is reduced by 70% and the computational effort required to decompose the matrix is reduced by approximately 97%.

B. Recomputing Residues after Elimination of Inaccurate Poles

The experimental study in Ref. 16 found two main problems with determining structural poles from time-domain realization algorithms.

1) Many structural poles converge only after massive overspecification of the model order [n_x in Eq. (14)]. Overspecification of model order, however, engenders additional computational or noise modes, which should not be retained for subsequent analysis using the identified model.

2) Poles that have converged can occasionally split into two nearly repeated (but nonphysical) modes as model-order overspecification is increased to converge other less observable poles.

In view of these pathologies, it is usually necessary to use one or several quantitative model quality indicators (MQI) to detect convergence and discriminate unwanted or unreliable modes from the system realization, e.g., see Ref. 13.

Unfortunately, because all of the global mode shapes are extracted through the realization process simultaneously, only the mode shapes of the full realization can be considered to be optimal in any sense. If, however, some modes are not retained for further analysis, the remaining mode shapes cannot be said to be optimal with respect to either the measured or the realized response parameters. This is often not a problem if the modes are well spaced and orthogonal via the measurement points. It can be a serious problem, however, when computational mode splitting, as just described, occurs in the realization analysis. In this case, the mode shape information can split between the two nearly identical poles. Hence, when splitting is detected and one pole is removed from the modal set, important mode shape information is also lost.

The application of the linear least-squares solution for C is straightforward in this case. Simply perform the system realization analysis to obtain A, B, and C in their decoupled modal form. Then, after computing various MQI and removing unreliable poles from A and B, C is recomputed using Eq. (18). It should be noted that the generalized controllability matrix $\tilde{W}_s$ must be recomputed using the reduced A and B matrices, rather than using the singular values and vectors of the Hankel matrix as in Eq. (21). This is not a significant computational burden, however, as powers of A are inexpensive to compute in the decoupled block modal form and the largest matrix inverse operation is of the order of the retained modes (which is relatively small in most instances).

C. Inclusion of Residual Flexibility Terms

One serious deficiency of the discrete-time state-space model form common to ERA and other algorithms is that it cannot always properly account for the residual flexibility effects of modes outside the measurement bandwidth. In particular, when using velocity sensors or accelerometers (arguably the most popular transducer types for modal testing), the modes above the measurement bandwidth contribute a sum term proportional to the Laplace terms s and s^2, respectively (see Refs. 13 and 17). When a discrete-time state-space model form is fit to this data, the residual flexibility effects are manifest as the sum total of additional real and imaginary poles. Depending on the magnitude of the residual effects, the identification of a large number of additional poles or identified states may require to accurately fit the experimental data. This is because these continuous Laplace terms cannot be expressed in terms of a finite number of discrete model poles.

We believe that there is a need to identify these residual effects in a physically meaningful way through estimation of residual flexibility terms, rather than artificially decomposed into numerous dynamic states with associated frequencies and damping rates that are not physically in the structure. One possible corrective approach is to compute a residual flexibility term by fitting the trend of the frequency-domain error between the measured frequency response function (FRF) and its model-based reconstruction. This approach is generally effective but ignores the weak coupling at all frequencies between the contribution of the identified modes and residues and the residual flexibility. The result also mixes least-squares time-domain and frequency-domain computations, obscuring the optimality criterion of the complete model response.

The present linear least-squares algorithm for estimating mode shapes of the system realization is easily extended to include the residual flexibility contribution in a consistent manner. Starting from the proper expression of the discrete FRF including residual terms, we have[13]

$$G(f_k) = C[\exp(j2\pi k/N)I - A]^{-1}B + D + (j2\pi k/N\Delta t)^p F \tag{25}$$

where F is the residual flexibility matrix and p is the differentiation order of the sensor type with respect to displacement, i.e., $p = 1$ for velocity and $p = 2$ for acceleration. Here the Laplace term s has been evaluated along the frequency axis $j\omega$ at the discrete frequency values $\omega_k = 2\pi f_k$, where $f_k = k/(N\Delta t)$ and k, N, and Δt are the sample index, total number of samples, and sampling rate, respectively.

Taking the inverse discrete Fourier transform (IDFT) of Eq. (25) leads to the following relationships:

$$\begin{aligned} M(0) &= D + M_{s^p}(0)F \\ M(1) &= CB + M_{s^p}(1)F \\ M(2) &= CAB + M_{s^p}(2)F \\ &\vdots \\ M(k) &= CA^{k-1}B + M_{s^p}(k)F \end{aligned} \tag{26}$$

where $M_{s^p}(i)$ are the real-valued IDFT coefficients of the discrete function

$$s_k^p = (j2\pi k/N\Delta t)^p \tag{27}$$

evaluated at

$$k = [1-(N/2) \quad \cdots \quad -1 \quad 0 \quad 1 \quad \cdots \quad (N/2)-1 \quad N/2] \tag{28}$$

The order of values for k given in Eq. (28) depends on the numerical algorithm for computing the IDFT; the given ordering is consistent with that used for the inverse fast Fourier transform in MatLab.[7] The time-domain coefficients $M_{s^p}(i)$ of the residual term are essentially normalized Markov parameters of the sum contribution of the modes above the measurement bandwidth to the estimated FRFs. Figure 1 shows the coefficients of s and s^2 for a small sample record with unit sample time. Using this result, we can then form the linear least-squares problem

$$\{M(1{:}s)\} = [C \quad F]\begin{bmatrix} \bar{W}_s \\ \{M_{s^p}(1{:}s)I_m\} \end{bmatrix} = [C \quad F]\hat{W} \tag{29}$$
$$M(0) = D + M_{s^p}(0)F$$

where I_m is the $m \times m$ identity matrix, m are the number of inputs, and

$$\{M_{s^p}(1{:}s)I_m\} = [M_{s^p}(1)I_m \quad M_{s^p}(2)I_m \quad \cdots \quad M_{s^p}(s)I_m] \tag{30}$$

The solution of Eq. (29) is then

$$[C \quad F] = M\{1{:}s\}\hat{W}^T(\hat{W}\hat{W}^T)^{-1}, \qquad D = M(0) - M_{s^p}(0)F \tag{31}$$

Note that when F and its coefficients are dropped from Eq. (29), the solutions for C and D are given by Eqs. (16) and (19), respectively.

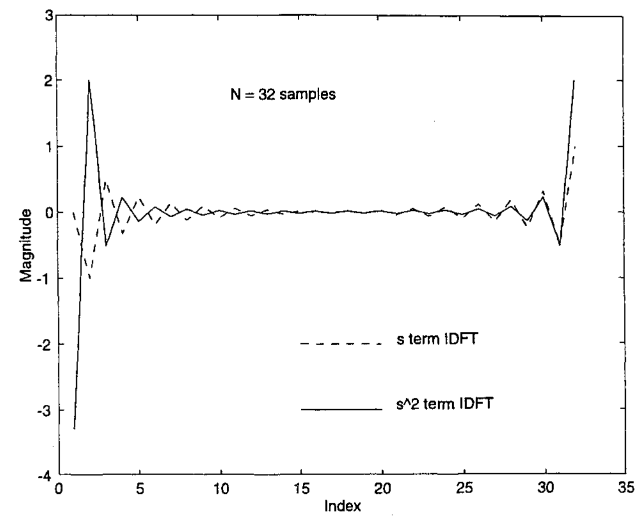

Fig. 1 Time-domain representation of the Laplace terms s and s^2.

The preceding formulation can often lead to an ill-conditioned matrix due to the mixture of continuous and discrete frequency-domain terms. To avoid these problems, we replace s_k in Eq. (27) and F in Eq. (29) by frequency-normalized counterparts

$$\bar{s}_k = s_k\Delta t = \frac{j2\pi k}{N}, \qquad \bar{F} = \frac{F}{(\Delta t)^p} \tag{32}$$

Then the estimated term $\bar{F}$ is multiplied by $(\Delta t)^p$ to obtain the correct residual flexibility term consistent with the continuous equations of motion.

V. Applications and Examples

A. Numerical Example Using Modal Test Simulator

To properly understand the behavior of any system identification algorithm, it is important to perform simulations using data that are characteristic of the actual data the algorithm will ultimately be applied to. Often, in time-domain system identification research, this realism is limited to the addition of Gaussian noise to Markov parameters of displacement outputs, which are generated in the time domain by the discretized system equations. Unfortunately, this approach neglects the process by which Markov parameters are usually obtained in testing; that is, frequency domain FRF estimation of acceleration data with signal conditioning, ensemble averaging, and digital signal processing. The signal processing and residual flexibility effects engendered can be far more significant on the performance of system realization algorithms than the level of noise that is typically encountered, at least in controlled modal testing environments. Therefore, a modal testing simulator was developed that includes all of the aforementioned effects, in addition to assumed measurement noise and burst random excitation.

Figure 2 shows a planar truss example. The model includes 36 unconstrained DOFs, 18 acceleration sensors and 3 externally applied force inputs; 3 of the sensors are collocated with the 3 inputs. The modal testing simulator was used to generate the FRFs for all 54 input–output pairings. The simulator used 8192 samples per ensemble, sampled at 1000 Hz with antialias filtering set at 400 Hz. The FRFs were obtained using 10 ensemble averages, and 1% noise was added to the measurements of the forces and accelerations.

The first stage of the time-domain system realization process consists of the estimation of the reliable system poles. For this example, the three driving-point, i.e., collocated output, measurements were retained as reference responses for a total of nine FRFs. An efficient form of ERA[15] was used with Hankel block dimensions of $r = 300$ and $s = 2000$ for total data matrix dimension of 900×6000. If all response measurements had been included in the data matrix, the dimension would have been 5400×6000. The order of the ERA-estimated model was varied from 50 to 100 states, and the convergence of various MQI were studied. The final model order chosen was $n_x = 52$, for a nominal set of 26 modes. Of those, three modes (six complex poles) were judged as inaccurate or unreliable and, thus, were removed from the modal set. A comparison of the retained modes to those of the exact model are shown in Table 1.

The second stage of the time-domain system identification consisted of using the present mode shape estimation algorithm to obtain mode shapes for the full set of 18 measured accelerometers. For the least-squares estimation, the first 2000 Markov parameters were used, consistent with the column dimension of the Hankel matrix used for the pole estimation. Figures 3 and 4 show the FRF reconstructions using the retained set of ERA modes together with the full mode shapes estimated using Eq. (18). Before proceeding, we make the following observations.

First, the transfer FRF shown in Fig. 4 cannot be obtained from the ERA analysis alone, as the Markov parameters for this input–output pair were not included in the Hankel matrix. That is, although the ERA-derived mode shapes did not include this response location, they were effectively estimated using the present procedure. Second, because we chose to eliminate three modes of the ERA realization, it was also useful to re-estimate the mode shapes for the reference point responses, as the ERA mode shapes were extracted simultaneously with the inaccurate modes. In this particular case, because the inaccurate poles were not closely coupled with any of the retained poles, the original ERA mode shapes and the re-estimated mode shapes at the reference points were essentially identical.

Finally, other than the resonance, which was not identified at approximately 312 Hz, the reconstructed FRFs are highly accurate at the resonance peaks. The zeros of the FRFs, however, show varying degrees of error, particularly the driving point response. These errors are due to the exclusion of a residual flexibility term in the ERA model and in the new mode shape estimation via Eq. (17). If, however, we include a term to model residual flexibility, as in Eq. (31), the reconstruction is significantly improved, as shown in Fig. 5. Further improvement at low frequency could possibly be obtained by changing the number of time points used, or by applying constraints of Eq. (25). The accuracy of the estimated mode shapes from Eqs. (18) and (31) with respect to the exact mode shapes of the example model is shown in the last two columns of Table 1, in terms of the modal assurance criteria (MAC) (normalized vector correlation). Whereas the mode shape estimate without including the residual term is nearly exact, there is an improvement in the mode shape by simultaneously estimating the residual flexibility. This result is consistent with the existence of a weak coupling between the two response contributions.

Table 1 Accuracy of identified modes using RP-ERA[a] with and without residual flexibility

Mode no.	f_{exact}, Hz	f_{ident}, Hz	Error, %	MAC using RP-ERA	MAC using RP-ERA with residual flexibility
1	23.062	23.065	0.0113	1.0000	1.0000
2	54.700	54.700	0.0011	1.0000	1.0000
3	81.566	81.566	0.0004	1.0000	1.0000
4	92.457	92.457	0.0003	1.0000	1.0000
5	132.26	132.26	0.0005	1.0000	1.0000
6	162.94	162.94	0.0028	0.9999	1.0000
7	171.20	171.20	0.0002	1.0000	1.0000
8	205.43	205.43	0.0002	1.0000	1.0000
9	235.15	235.15	0.0006	1.0000	1.0000
10	237.93	237.93	0.0003	1.0000	1.0000

[a]ERA using reference point responses.

B. Application to Experimental Data

Although the preceding numerical example was realistic via use of the modal testing simulator, it is often helpful to verify the accuracy accrued by the curve fitting procedure through its application

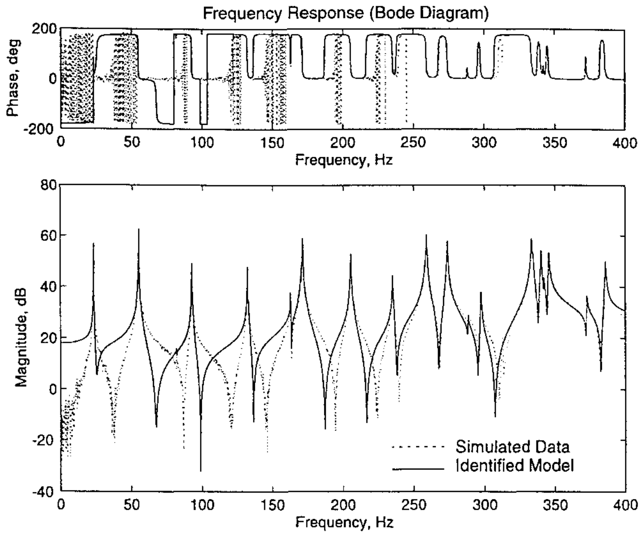

Fig. 3 Driving point FRF reconstruction.

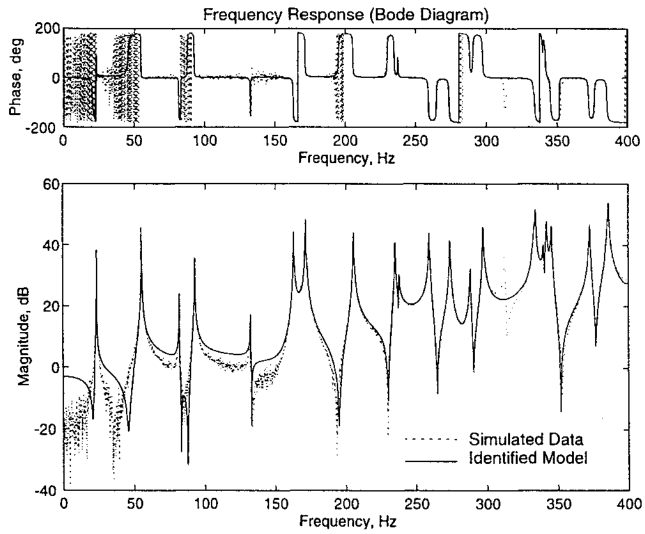

Fig. 4 Transfer FRF reconstruction.

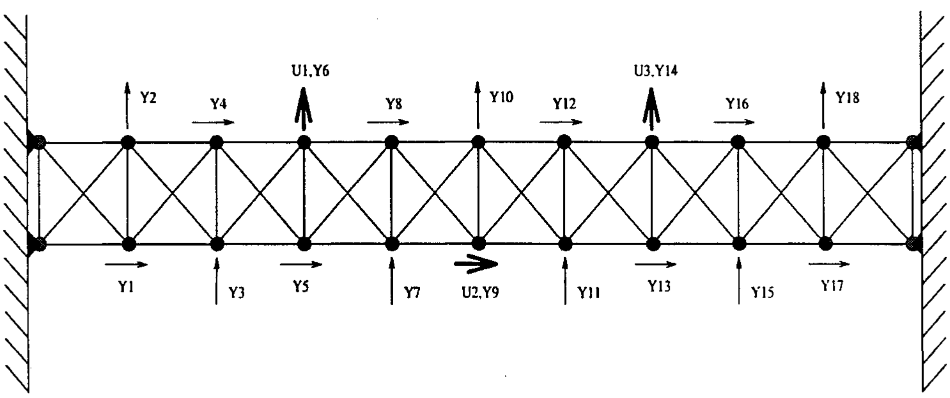

Fig. 2 Two-dimensional truss numerical example.

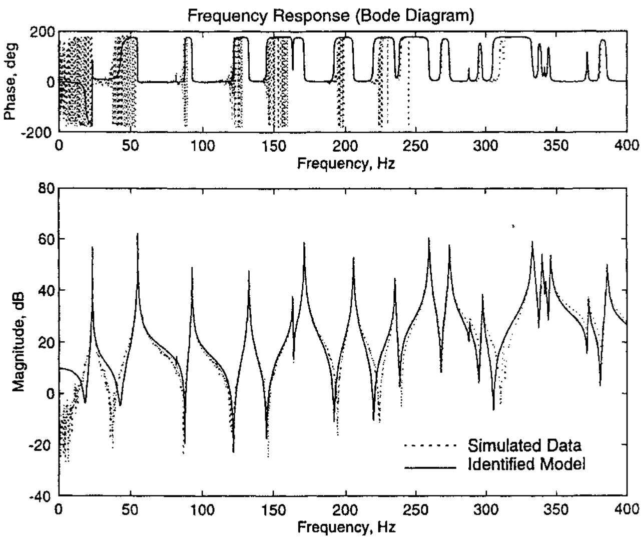

Fig. 5 Driving point FRF with residual flexibility.

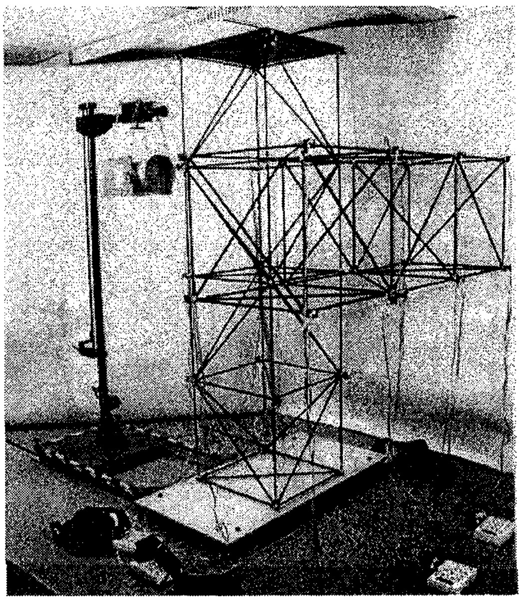

Fig. 6 Photograph of truss modal testing.

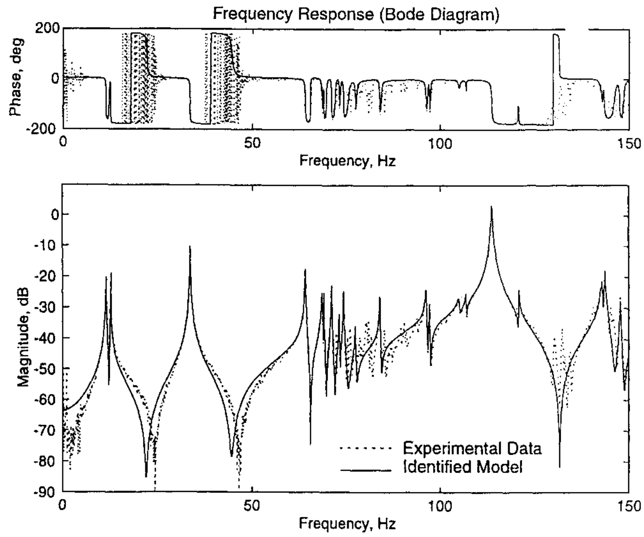

Fig. 7 Driving point FRF and reconstruction with residual flexibility for truss tower.

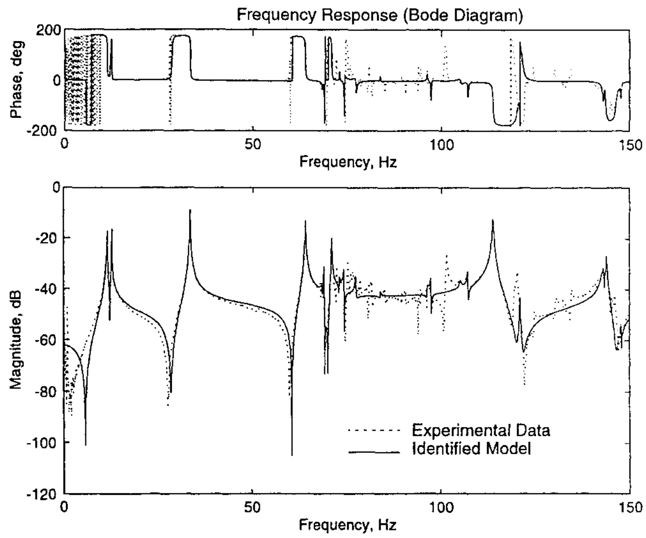

Fig. 8 Transfer FRF and reconstruction with residual flexibility for truss tower.

to actual experimentally measured data. Figure 6 shows a photograph of the three-dimensional cantilevered truss structure tested. The modal testing used one force input and 61 accelerometers (including a driving point location), with a sampling frequency of 500 Hz and 50 ensemble averages.

As in the numerical example, the poles were estimated using ERA with the single driving point measurement. After selecting $n_x = 100$ (50 modes), 28 modes were retained for the final model. The 61 response measurements were then re-estimated to yield the desired mode shapes and residual flexibility. The driving point FRF and a representative transfer FRF are shown in Figs. 7 and 8. One important lesson learned with this data was that it was important to include the last 20 times samples of the impulse response in the least-squares equation to obtain good estimates of the residual flexibility. This is because of the magnitude increase of $\boldsymbol{M}_{sp}$ as $k \rightarrow N$ in Fig. 1.

Generally, it is possible to obtain better curve fits in the frequency domain using techniques such as in Ref. 13, where frequencies can be selected near both resonances and antiresonances, or by using nonlinear estimators. It is also possible to obtain better frequency domain fits using the present time-domain approach by varying the amount of data used in the least-squares equation. Our purpose here was not to tune the algorithm for the best possible accuracy (which is likely to be problem dependent), but rather to show that, by simple modifications, a linear least-squares time-domain method can significantly improve the accuracy of the realized model in the frequency domain.

VI. Conclusions

A linear least-squares mode shape estimation algorithm using time-domain system realization theory has been presented. The present procedure augments existing time-domain system realization algorithms such as the ERA, ERA with data correlations, and Q-Markov covariance equivalent realization, by providing alternative techniques to compute (or re-estimate) global mode shapes when performing reference point-based pole estimation, or to re-estimate mode shapes after eliminating unreliable poles from the system realization. Furthermore, the procedure can be generalized to estimate residual flexibility terms that are not modeled in a physically meaningful way within the discrete state-space form. These capabilities have been demonstrated via numerical and experimental data.

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